INTEGRAL TAK TENTU (ITT)

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LOGO

DEFINISI & SIFAT ITT

$$\int f(x)dx = F(x) + C$$

Integrand — Fungsi primitif — konstanta integrasi

SIFAT-SIFAT ITT:

1. $\int kf(x)\,dx = k\int f(x)\,dx;\ k$: konstanta
2. $\int \{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx$

Beberapa rumus ITT

1. $\int 0\,dx = C$
2. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C;\ (n \neq -1)$
3. $\int \cos x\,dx = \sin x + C$
4. $\int \sin x\,dx = -\cos x + C$
5. $\int \sec^2 x\,dx = \operatorname{tg} x + C$
6. $\int \operatorname{cosec}^2 x\,dx = -\operatorname{cot} g\, x + C$
7. $\int \sec x \operatorname{tg} x\,dx = \sec x + C$
8. $\int \operatorname{cosec} x \operatorname{cot} g\, x\,dx = -\operatorname{cosec} x + C$
9. $\int e^x\,dx = e^x + C$
10. $\int \frac{1}{x}dx = \ln x + C$
11. $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$
12. $\int \frac{1}{1+x^2}dx = \operatorname{arctg} x + C$

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Contoh:

Selesaikan: $I = \int \frac{1}{x^3}dx$

Penyelesaian: $I = \int x^{-3}dx = \frac{1}{-3+1}x^{-3+1} + C = -\frac{1}{2}x^{-2} + C$

Selesaikan: $I = \int x(1-x^2)^5\,dx$

Penyelesaian:

Misal: $u = 1 - x^2 \rightarrow \frac{du}{dx} = -2x \rightarrow x\,dx = -\frac{1}{2}du$

$I = \int u^5\left(-\frac{1}{2}du\right) = -\frac{1}{2}.\frac{1}{6}u^6 + C = -\frac{1}{12}(1-x^2)^6 + C$

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Integrasi Parsial

$$\int u\,dv = uv - \int v\,du$$

Rumus ini sangat berguna teritama jika integrand terdiri dari fungsi-fungsi transcendent, misalnya: $\ln x$, $\arcsin x$, $\operatorname{arctg} x$, atau hasil ganda seperti: xe^x, $e^x \sin x$, $x\cos x$, $x^2 \ln x$.

Cara memakai rumus ini:

a. dv dipilih sehingga v mudah dicari

b. $\int v\,du$ harus menjadi lebih mudah daripada $\int u\,dv$.

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Contoh:

1. $I = \int xe^{-x}\,dx$

Misal: $u = x$ $\qquad dv = e^{-x}dx$

$du = dx$ $\qquad v = \int e^{-x}\,dx = -e^{-x}$

$$I = xe^{-x} - \int -e^{-x}\,dx = xe^{-x} - e^{-x} + C.$$

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2. $I = \int \text{arctg } x\,dx$

Misal: $u = \text{arctg } x$ $\qquad dv = dx$

$du = \dfrac{1}{1+x^2}$ $\qquad v = \int dx = x$

$$I = \int \text{arctg } x\,dx = x\,\text{arctg } x - \int x.\frac{1}{1+x^2}dx$$

$$= x\,\text{arctg } x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$$

$$= x\,\text{arctg } x - \frac{1}{2}\ln(1+x^2) + C$$

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Penerapan 1/D pada ITT

$D = \dfrac{d}{dx}$ operator derivatif/turunan

$\dfrac{1}{D} = D^{-1} = \int .....dx$ operator integral

Deret Maclaurin dari:

$$\frac{1}{1-D} = 1 + D + D^2 + ...$$

$$\frac{1}{1+D} = 1 - D + D^2 - ....$$

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$D\sin ax = a\cos ax; \quad D^2\sin ax = -a^2\sin ax$

$D\cos ax = -a\sin ax; \quad D^2\cos ax = -a^2\cos ax$

$$\left.\begin{aligned}(D^2-b^2)\sin ax &= (-a^2-b^2)\sin ax\\ (D^2-b^2)\cos ax &= (-a^2-b^2)\cos ax\end{aligned}\right\}$$ sehingga:

$$\frac{1}{D^2-b^2}\sin ax = \frac{\sin ax}{-a^2-b^2}; \qquad \frac{1}{D^2-b^2}\cos ax = \frac{\cos ax}{-a^2-b^2}.$$

RUMUS: $\frac{1}{D}e^{ax}V = e^{ax}\frac{1}{D+a}V$

$$\frac{1}{D}UV = U.\frac{1}{D}V - DU.\frac{1}{D^2}V + D^2U.\frac{1}{D^3}V - \ldots$$

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Contoh 1:

$$I = \int e^{3x}\left(x^2-3x+2\right)dx = \frac{1}{D}e^{3x}\left(x^2-3x+2\right) = e^{3x}\frac{1}{D+3}\left(x^2-3x+2\right)$$

$$= e^{3x}.\frac{1}{3}.\frac{1}{\left(1+\frac{D}{3}\right)}\left(x^2-3x+2\right) = \frac{1}{3}e^{3x}\left(1-\frac{D}{3}+\frac{D^2}{9}-\ldots\right)\left(x^2-3x+2\right)$$

$$= \frac{1}{3}e^{3x}\left(\left(x^2-3x+2\right)-\frac{1}{3}(2x-3)+\frac{1}{9}.2\right)+C = \frac{1}{3}e^{3x}\left(x^2-\frac{11}{3}x+\frac{29}{9}\right)+C$$

$$I = \frac{1}{27}e^{3x}\left(9x^2-33x+29\right)+C$$

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Contoh 2:

$$\int x^3 e^{-2x}\,dx = \frac{1}{D}x^3e^{-2x} = x^3.\frac{1}{D}e^{-2x} - Dx^3.\frac{1}{D^2}e^{-2x} + D^2x^3\frac{1}{D^3}e^{-2x} - D^3x^3\frac{1}{D^4}e^{-2x}$$

$$= -\frac{1}{2}x^3e^{-2x} - \frac{3}{4}x^2e^{-2x} - \frac{3}{4}xe^{-2x} - \frac{3}{8}e^{-2x} + C$$

$$= -\frac{1}{8}e^{-2x}\left(4x^3 + 6x^2 + 6x + 3\right) + C$$

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Rumus Reduksi

1. $\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx$; n : bilangan bulat positif ≥ 1.

2. $\int \sin^n x dx = \frac{-\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$; n : bilangan bulat positif ≥ 2.

3. $\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$; n : bilangan bulat positif ≥ 2.

4. $\int \mathrm{tg}^n x\,dx = \frac{\mathrm{tg}^{n-1}x}{n-1} - \frac{n-1}{n}\int \mathrm{tg}^{n-2}x\,dx$; n : bilangan bulat positif ≥ 2.

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5. $\int \text{cotg}^n x\,dx = \frac{-\text{cotg}^{n-1} x}{n-1} - \int \text{cotg}^{n-2} x\,dx$; n : bilangan bulat positif ≥ 2.

6. $\int \sec^n x\,dx = \frac{\sec^{n-2} x \,\text{tg}\, x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$; n : bilangan bulat positif ≥ 2.

7. $\int \text{cosec}^n x\,dx = \frac{-\text{cosec}^{n-2} x \,\text{cotg}\, x}{n-1} + \frac{n-2}{n-1}\int \text{cosec}^{n-2} x\,dx$; n : bilangan bulat positif ≥ 2

8. $\int \frac{dx}{(1+x^2)^n} = \frac{x}{2(n-1)(1+x^2)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{dx}{(1+x^2)^{n-1}}$; $n \neq 1$.

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Integrasi Fungsi Pecah Rasional

$$\int \frac{T(x)}{N(x)}\,dx$$

dengan derajat pembilang < derajat penyebut.

Ada beberapa kasus berhubungan dengan penyebut:

1. Jika $N(x) = (ax+b)(cx+d)$ maka

$$\frac{T(x)}{N(x)} = \frac{A}{ax+b} + \frac{B}{cx+d}$$

2. Jika $N(x) = (ax+b)^k (cx+d)$ maka

$$\frac{T(x)}{N(x)} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \ldots + \frac{K}{(ax+b)^k} + \frac{U}{cx+d}$$

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3. Jika $N(x)=(ax^2+bx+c)(px+q)$ dengan $D=(b^2-4ac)<0$ maka

$$\frac{T(x)}{N(x)}=\frac{Ax+B}{ax^2+bx+c}+\frac{C}{px+q}$$

4. Jika $N(x)=(ax^2+bx+c)^2(px+q)$ dengan $D=(b^2-4ac)<0$ maka

$$\frac{T(x)}{N(x)}=\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{(ax^2+bx+c)^2}+\ldots+\frac{Ux+V}{(ax^2+bx+c)^k}+\frac{W}{px+q}$$

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Contoh 1:

$$I=\int\frac{23-2x}{2x^2+9x-5}dx=\int\frac{23-2x}{(2x-1)(x+5)}dx$$

$\frac{23-2x}{(2x-1)(x+5)}=\frac{A}{2x-1}+\frac{B}{x+5}$ dikalikan dengan $(2x-1)(x+5)$, menjadi

$$23-2x=A(x+5)+B(2x-1)$$

Untuk $x=-5\Rightarrow 23-2(-5)=B(-11)\Rightarrow B=-3$

Untuk $x=\frac{1}{2}\Rightarrow 23-2\left(\frac{1}{2}\right)=A\left(\frac{11}{2}\right)\Rightarrow A=4$

$$I=\int\frac{4}{2x-1}dx+\int\frac{-3}{x+5}dx=2\ln|2x-1|-3\ln|x+5|+C$$

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Contoh 2:

$$I = \int \frac{3x+1}{x^2+2x+1}\,dx = \int \frac{3x+1}{(x+1)^2}\,dx$$

$\frac{3x+1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$ dikalikan dengan $(x+1)^2$, menjadi

$3x+1 = A(x+1)+B$

Untuk $x=-1 \Rightarrow -3+1 = B \Rightarrow B=-2$

Untuk $x=0 \Rightarrow 1 = A+B \Rightarrow A=3$

$$I = \int \frac{3}{x+1}\,dx + \int \frac{-2}{(x+1)^2}\,dx = 3\ln|x+1| + \frac{2}{x+1} + C$$

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Contoh 3:

$$I = \int \frac{x-1}{(x+1)(x^2+1)}\,dx$$

$\frac{x-1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$ dikalikan $(x+1)(x^2+1)$, menjadi

$x-1 = A(x^2+1)+(Bx+C)(x+1)$

Ketemu $A=-1,\ B=1,\ C=0$

$$I = \int \frac{-1}{x+1}\,dx + \int \frac{x}{x^2+1}\,dx = -\ln|x+1| + \frac{1}{2}\ln|x^2+1| + C$$

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Contoh 4:

$$I = \int \frac{2x^2+3}{(x^2+1)^2}\,dx$$

$\frac{2x^2+3}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2}$ dikalikan dengan $(x^2+1)^2$, menjadi

$$2x^2+3 = (Ax+B)(x^2+1)+Cx+D$$

Untuk

$$\left.\begin{array}{r} x=0 \Rightarrow 3 = B+D \\ x=1 \Rightarrow 5 = 2A+2B+C+D \\ x=-1 \Rightarrow 5 = -2A+2B-C+D \\ x=2 \Rightarrow 11 = 10A+5B+2C+D \end{array}\right\} \Rightarrow B=2, D=1, A=0, C=0.$$

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$$I = \int \left[\frac{2}{x^2+1} + \frac{1}{(x^2+1)^2}\right] dx = 2\int \frac{1}{1+x^2}\,dx + \int \frac{1}{(1+x^2)^2}\,dx$$

$$= 2\,\text{arctg}\,x + \left\{\frac{x}{2(1+x^2)} + \frac{1}{2}\,\text{arctg}\,x\right\} + C$$

$$= \frac{5}{2}\,\text{arctg}\,x + \frac{x}{2(1+x^2)} + C$$

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Integrasi Fungsi Trigonometri

Ingat-ingat beberapa rumus berikut:

1. $\sin^2 x + \cos^2 x = 1$
2. $\sin 2x = 2\sin x \cos x$
3. $\cos 2x = \cos^2 x - \sin^2 x$
4. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
5. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
6. $\sin(-x) = -\sin x;\ \cos(-x) = \cos x$
7. $\sin \alpha x \sin \beta x = \frac{1}{2}[\sin(\alpha + \beta)x + \sin(\alpha - \beta)x]$
8. $\cos \alpha x \cos \beta x = \frac{1}{2}[\cos(\alpha + \beta)x + \cos(\alpha - \beta)x]$
9. $\sin \alpha x \sin \beta x = \frac{1}{2}[\cos(\alpha - \beta)x - \cos(\alpha + \beta)x]$

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Bentuk: $\int \sin \alpha x \cos \beta x\, dx;\ \int \cos \alpha x \cos \beta x\, dx;\ \int \sin \alpha x \sin \beta x\, dx$

1. $\int \sin 6x \cos 2x\, dx = \int \frac{1}{2}[\sin 8x + \sin 4x]dx = \frac{1}{2}\left[-\frac{1}{8}\cos 8x - \frac{1}{4}\cos 4x\right] + C$
2. $\int \cos 6x \cos 3x\, dx = \int \frac{1}{2}[\cos 9x + \cos 3x]dx = \frac{1}{2}\left[\frac{1}{9}\sin 9x + \frac{1}{3}\sin 3x\right] + C$
3. $\int \cos^2 2x\, dx = \int \frac{1}{2}[1 + \cos 4x]dx = \frac{1}{2}\left[x + \frac{1}{4}\sin 4x\right] + C$
4. $\int \sin 3x \sin 2x\, dx = \int \frac{1}{2}[\cos x - \cos 5x]dx = \frac{1}{2}\left[\sin x - \frac{1}{5}\sin 5x\right] + C$

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Bentuk: $\int R(\sin x, \cos x)dx;\ \ R = \text{fs rasional}$

Substitusi: $\text{tg}\,\frac{x}{2} = t \Rightarrow \frac{x}{2} = \text{arctg}\, t, \quad x = 2\ \text{arctg}\, t \Rightarrow dx = \frac{2dt}{1+t^2}$

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{2\,\text{tg}\,\frac{x}{2}}{1+\text{tg}^2\frac{x}{2}} \Rightarrow \sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{1-\text{tg}^2\frac{x}{2}}{1+\text{tg}^2\frac{x}{2}} \Rightarrow \cos x = \frac{1-t^2}{1+t^2}$$

Maka:

$$\int R(\sin x, \cos x)dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}.$$

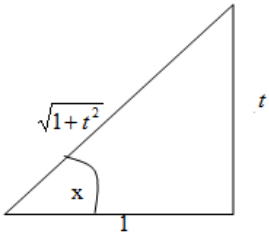
Contoh:

$$\int \frac{dx}{5+4\cos x} = \int \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)}\cdot\frac{2dt}{1+t^2} = 2\int \frac{dt}{9+t^2} = \frac{2}{3}\text{arctg}\,\frac{t}{3}+C$$

$$= \frac{2}{3}\text{arctg}\left(\frac{1}{3}\text{tg}\,\frac{x}{2}\right)+C$$

Bentuk: $\int R(\sin x, \cos x)dx = \int R(-\sin x, -\cos x)dx$

Substitusi: $\text{tg}\, x = t \Rightarrow x = \text{arctg}\, t \Rightarrow dx = \dfrac{dt}{1+t^2}$



$$\sin x = \frac{t}{\sqrt{1+t^2}}$$
$$\cos x = \frac{1}{\sqrt{1+t^2}}$$

CONTOH:

$$I = \int \frac{1}{\sin^2 x - \sin x \cos x}dx = \int \frac{1}{\left(\frac{t}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)} \cdot \frac{dt}{1+t^2}$$

$$= \int \frac{dt}{t(t-1)} = \int \left(\frac{1}{t-1} - \frac{1}{t}\right)dt = \ln\left|\frac{t-1}{t}\right| + C = \ln\left|1 - \cot g\, x\right| + C$$

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Bentuk: $\int R(\text{tg}\, x)dx$

Substitusi: $\text{tg}\, x = t \Rightarrow x = \text{arctg}\, t \Rightarrow dx = \dfrac{dt}{1+t^2}$

CONTOH:

$$I = \int \frac{1+\text{tg}\, x}{1-\text{tg}\, x}dx = \int \frac{1+t}{1-t} \cdot \frac{dt}{1+t^2} = \int \frac{1+t}{(1-t)(1+t^2)}dt$$

$\dfrac{1+t}{(1-t)(1+t^2)} = \dfrac{A}{1-t} + \dfrac{Bt+C}{1+t^2}$ dikalikan dengan $(1-t)(1+t^2)$, maka:

$$1+t = A(1+t^2) + (Bt+C)(1-t)$$

$$1+t = (A+C) + (B-C)t + (A-B)t^2$$

$$\left.\begin{aligned} A+C &= 1 \\ B-C &= 1 \\ A-B &= 0 \end{aligned}\right\} \Rightarrow A = 1,\ B = 1, C = 0.$$

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$$I = \int \left[\frac{1}{1-t} + \frac{t}{1+t^2} \right] dt = -\ln|1-t| + \frac{1}{2}\ln|1+t^2| + C = -\ln\left|\frac{1-t}{\sqrt{1+t^2}}\right| + C$$

$$= -\ln\left|\frac{1-\operatorname{tg} x}{\sec x}\right| + C = -\ln\left|(1-\operatorname{tg} x)\cos x\right| + C = -\ln|\cos x - \sin x| + C$$

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Integrasi dgn substitusi trigonometri

No	Integrand	Substitusi	Hasil:
1	$\int \frac{dx}{\sqrt{a^2-x^2}}$	$x = a\sin t$	$\arcsin\frac{x}{a} + C$
2	$\int \sqrt{a^2-x^2}\,dx$		$\frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2 \arcsin\frac{x}{a}\right) + C$
3	$\int \frac{dx}{a^2+x^2}$	$x = a\operatorname{tg} t$	$\frac{1}{a}\operatorname{arctg}\frac{x}{a} + C$
4	$\int \frac{dx}{\sqrt{a^2+x^2}}$		$\ln\left\|x + \sqrt{x^2+a^2}\right\| + C$

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5	$\int \sqrt{x^2+a^2}\,dx$		$\frac{1}{2}\left(x\sqrt{x^2+a^2}+a^2\ln\left\|x+\sqrt{x^2+a^2}\right\|\right)+C$
6	$\int \frac{dx}{\sqrt{x^2-a^2}}$	$x=a\sec t$	$\ln\left\|x+\sqrt{x^2-a^2}\right\|+C$
7	$\int \sqrt{x^2-a^2}\,dx$		$\frac{1}{2}\left(x\sqrt{x^2-a^2}-a^2\ln\left\|x+\sqrt{x^2-a^2}\right\|\right)+C$

1. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{a\cos t}.a\cos t\,dt = \int dt = t+C = \arcsin\frac{x}{a}+C$

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