LOGO

TURUNAN FUNGSI

Anita T. Kurniawati

TURUNAN FUNGSI

Definisi:

Turunan dari fungsi $y = f(x)$ terhadap x adalah:

$$y' = \frac{dy}{dx} = Dy = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

. **SIFAT-SIFAT TURUNAN**

$u = u(x),\ v = v(x), C =$ konstanta

1. $y = u \pm v \rightarrow y' = u' \pm v'$
2. $y = uv \rightarrow y' = u'v + uv'$
3. $y = Cv \rightarrow y' = Cv'$
4. $y = \frac{u}{v} \rightarrow y' = \frac{u'v - uv'}{v^2}$

LOGO

Rumus Turunan:

a. $y = C \rightarrow y' = 0$

b. $y = x^n \rightarrow y' = nx^{n-1}$

c. $y = \sin x \rightarrow y' = \cos x$

d. $y = \cos x \rightarrow y' = -\sin x$

e. $y = tg\, x \rightarrow y' = \sec^2 x$

f. $y = \cot g\, x \rightarrow y' = -\cos ec^2 x$

g. $y = \sec x \rightarrow y' = \sec x \,\mathrm{tg}\, x$

h. $y = \cos ec\, x \rightarrow y' = -\cos ec\, x \cot g\, x$

i. $y = e^x \rightarrow y' = e^x$

j. $y = \ln x \rightarrow y' = \frac{1}{x}$

k. $y = \arcsin x \rightarrow y' = \frac{1}{\sqrt{1-x^2}}$

l. $y = \mathrm{arctg}\, x \rightarrow y' = \frac{1}{1+x^2}$

m. $y = \sinh x \rightarrow y' = \cosh x$

LOGO

Contoh:

Dapatkan y' dari $y = \frac{1}{x^3}$

Penyelesaian: $y = x^{-3} \rightarrow y' = -3x^{-3-1} = -3x^{-4} = \frac{-3}{x^4}$

LOGO

Aturan Berantai:

Jika $y = f(u), g(x) \rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Dapatkan y' dari $y = (2x^3 - 4x + 9)^4$

Penyelesaian:

Misal: $u = 2x^3 - 4x + 9 \rightarrow \frac{du}{dx} = 6x^2 - 4$; $\quad y = u^4 \rightarrow \frac{dy}{du} = 4u^3$

Aturan Berantai (AB): $y' = \frac{dy}{du}\frac{du}{dx} = 4u^3(6x^2 - 4) = 4(6x^2 - 4)(2x^3 - 4x + 9)^3$.

LOGO

Turunan Tingkat Tinggi

Dari $y = f(x)$ maka:

$y' = f'(x) = \frac{dy}{dx} = Dy$ menyatakan turunan pertama

$y'' = f''(x) = \frac{d^2y}{dx^2} = D^2y$ menyatakan turunan kedua

....

$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n} = D^ny$ menyatakan turunan tingkat n.

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$$

LOGO

Rumus Leibnitz

$$D^n(UV) = UD^nV + nDU.D^{n-1}V + \frac{1}{2!}n(n-1)D^2U.D^{n-2}V + ...$$

CONTOH:

Dapatkan $y^{(n)}$ dari $y = x^2e^x$

Penyelesaian:

Misal: $u = x^2; \quad v = e^x$

$Du = 2x, D^2u = 2, D^3u = 0; \quad Dv = e^x, D^2v = e^x; D^nv = e^x$

$$D^n(x^2e^x) = x^2e^x + n(2x)e^x + \frac{1}{2}n(n-1)(2)e^x + 0 = e^x(x^2 + 2nx + n^2 - n).$$

LOGO

Limit bentuk tak tentu

Bentuk-bentuk tak tentu: $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0.\infty, 1^\infty, \infty^0$.

Penyelesaian limit bentuk tak tentu:

I. Bentuk $\frac{0}{0}$. Berlaku aturan L'Hospital.

Aturan L'Hospital:

Jika $f(a) = f'(a) = f''(a) = ... = f^{(n-1)}(a) = 0$ dan $g(a) = g'(a) = g'(a) = ... = g^{(n-1)}(a) = 0$ tetapi satu (masing-masing) dari $f^{(n)}(a)$ dan $g^{(n)}(a)$ tidak nol, maka

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}.$$

II. Bentuk $\frac{\infty}{\infty}$. Berlaku langsung aturan L'Hospital.

LOGO

III. Bentuk $\infty - \infty$:

$$\lim_{x\to a}\{f(x)-g(x)\} = \lim_{x\to a}\left\{\frac{1}{\left(\frac{1}{f(x)}\right)} - \frac{1}{\left(\frac{1}{g(x)}\right)}\right\} = \lim_{x\to a}\frac{\frac{1}{g(x)}-\frac{1}{f(x)}}{\frac{1}{f(x)g(x)}} = \frac{0}{0}, \text{dst.}$$

IV. Bentuk $0.\infty$:

$$\lim_{x\to a} f(x).g(x) = \lim_{x\to a}\frac{f(x)}{\left(\frac{1}{g(x)}\right)} = \frac{0}{0} \text{ atau } \lim_{x\to a} f(x).g(x) = \lim_{x\to a}\frac{g(x)}{\left(\frac{1}{f(x)}\right)} = \frac{\infty}{\infty}, \text{dst.}$$

V. Bentuk $1^{\infty}, \infty^{0}, 0^{0}$:

$$\lim_{x\to a} f(x)^{g(x)} = e^{\lim\limits_{x\to a} g(x)\ln f(x)}$$

LOGO

Contoh:

1. $\lim\limits_{x\to 1}\frac{x^2-1}{x^3-1} = \overbrace{\left(\frac{1^2-1}{1^3-1}=\frac{0}{0}\right)}^{cek:} = \lim\limits_{x\to 1}\frac{2x}{3x^2} = \frac{2(1)}{3(1^2)} = \frac{2}{3}$

2. $\lim\limits_{x\to 0^+}\frac{\ln(\sin x)}{\ln(\text{tg}\, x)} = \overbrace{\left(\frac{\ln(\sin 0)}{\ln(\text{tg}\, x)}=\frac{\ln 0^+}{\ln 0^+}=\frac{-\infty}{-\infty}=\frac{\infty}{\infty}\right)}^{cek:} = \lim\limits_{x\to 0^+}\frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{\sec^2 x}{\text{tg}\, x}\right)} = \lim\limits_{x\to 0^+}\frac{\cot g\, x.\text{tg}\, x}{\sec^2 x} = 1$

3. $\lim\limits_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) = \overbrace{\left(\frac{1}{0^+}-\frac{1}{e^0-1}=\frac{1}{0}-\frac{1}{0}=\infty-\infty\right)}^{cek:} = \lim\limits_{x\to 0^+}\frac{(e^x-1)-x}{x(e^x-1)}$

$$= \overbrace{\left(\frac{(e^0-1)-0}{0(e^0-1)}=\frac{0}{0}\right)}^{cek:} = \lim_{x\to 0^+}\frac{e^x-1}{1(e^x-1)+xe^x} = \overbrace{\left(\frac{e^0-1}{1(e^0-1)+0.e^0}=\frac{0}{0}\right)}^{cek:}$$

$$= \lim_{x\to 0^+}\frac{e^x}{e^x+e^x+xe^x} = \frac{1}{2}$$

LOGO

$$4.\ \lim_{x\to 0} x\operatorname{cosec} x = \overbrace{0\operatorname{cosec}x = 0.\infty}^{\text{cek:}} = \lim_{x\to 0} x.\frac{1}{\sin x} = \lim_{x\to 0}\frac{x}{\sin x}$$

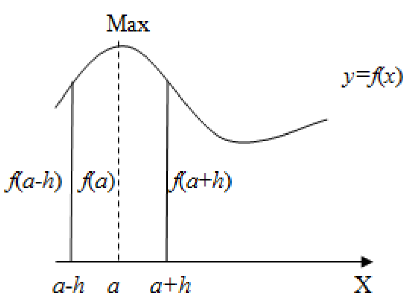
$$= \overbrace{\left(\frac{0}{\sin 0} = \frac{0}{0}\right)}^{\text{cek:}} = \lim_{x\to 0}\frac{1}{\cos 0} = 1.$$

$$5.\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \overbrace{\left(1^{\infty}\right)}^{\text{cek:}} = e^{\lim\limits_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)} = \overbrace{e^{\infty.0}}^{\text{cek:}} = e^{\lim\limits_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}} = \overbrace{\left(e^{\frac{0}{0}}\right)}^{\text{cek:}} = e^{\lim\limits_{x\to\infty}\frac{\frac{\frac{-1}{x^2}}{1+\frac{1}{x}}}{\frac{-1}{x^2}}}$$

$$= e^{\lim\limits_{x\to\infty}\frac{1}{1+\frac{1}{x}}} = e$$

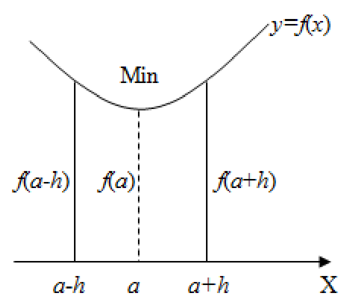
LOGO

NILAI EXTRIM



$y=f(x)$ max di $x = a$:

$$\begin{cases} f(a-h) < f(a) \\ f(a+h) < f(a) \end{cases}$$

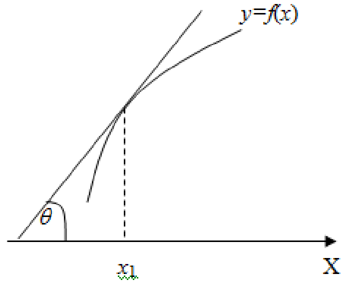


$y=f(x)$ min di $x = a$:

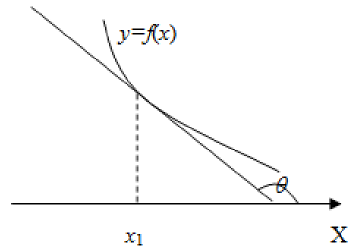
$$\begin{cases} f(a-h) > f(a) \\ f(a+h) > f(a) \end{cases}$$

LOGO

FUNGSI NAIK DAN TURUN

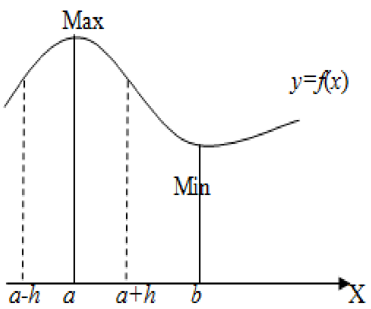


f naik di x_1 jika $f'(x_1) > 0$



f turun di x_1 jika $f'(x_1) < 0$

LOGO



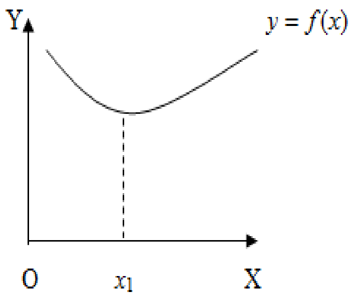
$$\left.\begin{array}{r} f'(a-h) < 0 \\ f'(a) = 0 \\ f'(a+h) > 0 \end{array}\right\} f(b) \text{ min } \& \ f''(b) > 0 .$$

Titik pada $y=f(x)$ dimana garis singgungnya mendatar (// sb. X) disebut titik kritis.

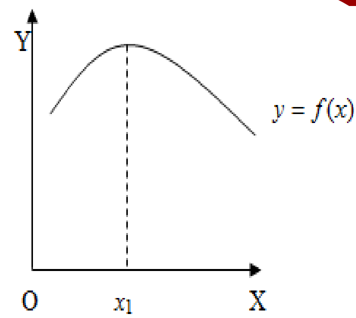
$$\left.\begin{array}{r} f'(a-h) > 0 \\ f'(a) = 0 \\ f'(a+h) < 0 \end{array}\right\} f(a) \text{ max } \& \ f''(a) < 0$$

LOGO

Cekung ke atas dan ke bawah



Cekung ke atas x_1, $f''(x_1) > 0$



Cekung ke bawah di x_1, $f''(x_1) < 0$

LOGO

Nilai Extrim (max dan min) dari *f*(*x*):

Dicari dulu $f'(x)$ dan $f''(x)$.

Syarat titik kritis (titik stasioner): $f'(x) = 0$. Misal ketemu titik kritis: $x = a$.

Jika $f''(a) < 0 \Rightarrow Y_{max} = f(a)$

Jika $f''(a) > 0 \Rightarrow Y_{min} = f(a)$

Titik belok dicari dari $f''(a) = 0$.

LOGO

Contoh:

Dapatkan titik-titik maximum dan minimum, titik belok dan sket grafik dari: $f(x) = 2x^3 - 24x + 5$.

Penyelesaian:

$f(x) = 2x^3 - 24x + 5$

$f'(x) = 6x^2 - 24$

$f''(x) = 12x$

$f'''(x) = 12$

Syarat extrim:

$f'(x) = 0 \rightarrow 6x^2 - 24 = 0 \rightarrow x^2 - 4 = 0 \rightarrow (x+2)(x-2) = 0 \rightarrow x_1 = -2,\ x_2 = 2$

LOGO

Untuk $x_1 = -2 \Rightarrow f''(-2) = 12(-2) = -24 < 0 \Rightarrow Y_{max}$

$Y_{max} = f(-2) = 2(-2)^3 - 24(-2) + 5 = 37$

Koordinat titik maximum di A(-2, 37).

Untuk $x_2 = 2 \Rightarrow f''(2) = 12(2) = 24 > 0 \Rightarrow Y_{min}$

$Y_{min} = f(2) = 2(2)^3 - 24(2) + 5 = -27$

Koordinat titik minimum di B(2, -27).

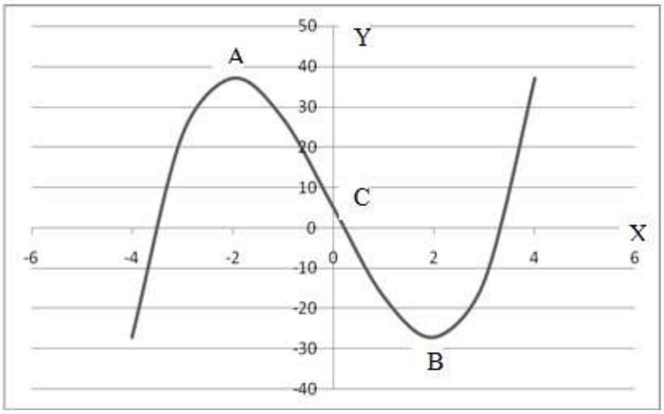
Koordinat titik belok didapat dari $f''(x) = 0 \Rightarrow 12x = 0 \rightarrow x = 0$.

$y = f(0) = 2(0) - 24(0) + 5 = 5$

Koordinat titik belok di C(0, 5).

LOGO

Grafik:



LOGO